On Abelian Bosonization of Free Fermi Fields in Three Space Dimensions

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Abstract. One of the methods used to extend two-dimensional bosonization to four space-time dimensions involves a transformation to new spatial variables so that only one of them appears kinematically. The problem is then reduced to an Abelian version of two-dimensional bosonization with extra "internal" coordinates. On a formal level, putting these internal coordinates on a finite lattice seems to provide a well-defined prescription for calculating correlation functions. However, in the infinite-lattice or continuum limits, certain difficulties appear that require very delicate specification of all of the many limiting procedures involved in the construction.

Introduction

After the initial success of bosonization in 1+1 dimensions [1,2] Alan Luther presented a heuristic formula [3] for bosonization of free, massless relativistic fermions in 3+1 dimensions. A configuration-space transform [4] was later used by H. Aratyn [5] to put this into a prettier but no less heuristic form. For other approaches to this problem the reader is referred to References [6] and [7].

The basic idea of the method described here is to view field theories in 3+1 dimensions as 1+1 dimensional theories with an internal, non-kinetic degree of freedom. The main subject of this paper is to examine carefully the limiting procedures involved in bosonizing a 1+1 dimensional free fermion with an internal degree of freedom that takes values that are in a continuum.

Tomographic transform

We begin by reviewing the transformation to a 1+1 dimensional theory both for bosons and for fermions. We then present the standard Abelian technique for bosonizing the 1+1 dimensional fermion when the internal degrees of freedom lie on a finite lattice. There follows a careful analysis of the continuum limit of this lattice in the context of a simplified model.

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The tomographic transform of a free scalar field $\phi(\mathbf{x}, t)$ with mass m in 3+1 dimensions is defined as

$$\tilde{\phi}(y, \mathbf{n}, t) = \frac{1}{2\pi} \int d^3\mathbf{x} \, \partial_y \delta(y - \mathbf{n} \cdot \mathbf{x}) \phi(\mathbf{x}, t).$$

Here $\mathbf{n}$ is a unit vector and $d^2\mathbf{n}$ an element of solid angle in the direction of $\mathbf{n}$.

The field equation satisfied by $\tilde{\phi}(y, \mathbf{n}, t)$ and its equal time commutation relations are essentially those of a 1+1 dimensional scalar field, also free and with mass m.

The corresponding transform for a fermion field is

$$\tilde{\psi}^a(y, \mathbf{n}, t) = \frac{1}{2\pi} \int d^3\mathbf{x} \, \partial_y \delta(y - \mathbf{n} \cdot \mathbf{x}) \sum_\alpha u_\alpha^{\dagger a}(\mathbf{n}) \psi_\alpha(\mathbf{x}, t).$$

It satisfies the 1+1 dimensional Dirac equation for a right-moving fermion field, (that is, depending only on $y - t$) and has the expected equal-time anticommutation relations. Here $u^a(\mathbf{n})$, $a = 1, 2$, are orthonormal four-component spinors that satisfy $(\boldsymbol{\alpha} \cdot \mathbf{n}) u^a(\mathbf{n}) = u^a(\mathbf{n})$ where $\boldsymbol{\alpha}$ are the Dirac alpha matrices.

The strategy is to use standard Abelian 1+1 dimensional bosonization to construct the right-moving $\tilde{\psi}^a(y, \mathbf{n}, t)$, in terms of the right-moving part of the transformed boson $\tilde{\phi}(y, \mathbf{n}, t)$, assuming, to begin with, that $\mathbf{n}$ is restricted to a lattice. This is given formally, with the t dependence suppressed, as

$$\tilde{\phi}_r^a(y, \mathbf{n}) = \tfrac{1}{2}[\tilde{\phi}^a(y, \mathbf{n}) - \tfrac{1}{2} \int_{-\infty}^{\infty} dy' \, \epsilon(y - y') \, \partial_t \tilde{\phi}^a(y', \mathbf{n})],$$

where $\epsilon(y - y') = (y - y')/|y - y'|$. It follows that

$$\langle 0 | \tilde{\phi}_r^a(y, \mathbf{n}) \tilde{\phi}_r^b(y', \mathbf{n}') | 0 \rangle = -\frac{1}{4\pi} \delta^{ab} \delta(\mathbf{n}, \mathbf{n}') \ln\left(\mu[\alpha - i(y - y')]\right)$$

where μ and α are infrared and ultraviolet cutoffs, respectively.

We first take $\mathbf{n}$ to be discretely distributed on some lattice of directions and reinterpret the Dirac delta function as a Kronecker delta function. Replacing the pair a and $\mathbf{n}$ by a single index A, we find the formal bosonization expression

$$\tilde{\psi}^A(y) = \frac{1}{\sqrt{2\pi\alpha}} e^{i\tilde{\phi}_r^A(y)} K^A$$

where

$$K^A = \exp \frac{i}{2} \sqrt{\pi} \sum_C \epsilon^{AC} [\phi_r^C(\infty) - \phi_r^C(-\infty)]$$

and where $\epsilon^{AB} = -\epsilon^{BA}$, with $|\epsilon^{AB}|^2 = 1$. The Klein factors K^A are needed to make sure that fermion fields with different indices anticommute rather than commute. In what follows we will ignore these factors. Their only effect is to correct an occasional sign to its proper value.

The correlation functions for the fermion field are determined from the matrix elements of their bosonic representation in the bosonic vacuum. The two-point function is computed as

$$\langle 0|\tilde{\psi}^A(y)\tilde{\psi}^{\dagger B}(y')|0\rangle = \frac{1}{2\pi}\frac{\mu}{\{\mu[\alpha + i(y-y')]\}^{\delta^{AB}}}.$$

If $A = B$ we get the known fermion function. If $A \neq B$ then this vanishes in the limit $\mu \to 0$, so that the result is indeed proportional to δ^{AB}. When reinterpreted in terms of a continuous $\mathbf{n}$ we do get the correct two-point function for the transformed fermion field:

$$\langle 0|\tilde{\psi}^a(y,\mathbf{n})\tilde{\psi}^{\dagger b}(y',\mathbf{n}')|0\rangle = \frac{1}{2\pi}\frac{\delta^{ab}\delta(\mathbf{n},\mathbf{n}')}{\alpha - i(y-y')}.$$

All of the other fermion correlation functions come out properly as well.

To test the consistency of the bosonization formula we must consider the important fermion bilinear operators in the 3+1 dimensional theory such as the Poincaré group generators and the chiral charge operators. If Ω is one of these, does the sequence $\Omega[\psi_{3+1}] \leftrightarrow \Omega[\tilde{\psi}_{1+1}] \leftrightarrow \Omega[\tilde{\phi}_{1+1}] \leftrightarrow \Omega[\phi_{3+1}]$ make sense? Trouble arises when we consider the rotation and boost generators. The proper treatment of the continuum limit of the lattice plays a fundamental role here.

Simplified model and smoothing functions

In order to get at the heart of the problem of the continuum limit we will treat a simple case in which the the internal variable is one-dimensional. We thus consider a single component 1+1 dimensional massless chiral fermion field $\psi(x,u)$ that depends on a continuous internal variable u whose domain is the real line. The bosonization of the fermion field in the Dirac equation $(\partial_t + \partial_x)\psi(x,u) = 0$ will be in terms of a right-moving chiral massless boson field $\phi_r(x,u)$. Assuming for the moment that u is a discrete variable, we write, up to Klein factors, $\psi(x,u) = (2\pi\alpha)^{-1/2}\exp[i\sqrt{4\pi}\phi_r(x,u)]$ and obtain for the fermion two-point function

$$\langle 0|\psi(x,u)\psi^\dagger(x',u')|0\rangle = \frac{1}{2\pi}\frac{\delta_{u,u'}}{\alpha - i(x-x')}$$

For the bosonization procedure to work, it is clear that the exponent that appears in the evaluation of the fermion two-point function must be associated with a Kronecker delta function, rather than a Dirac delta function so that the logarithmic singularity in the bosonic two-point function exponentiates to a simple pole or a simple zero. We need a more analytic method of converting one type delta function to the other. Returning to a continuous u, we introduce a new boson field $\Phi(x,u) = \int du' f(u-u')\phi_r(x,u')$. We take $f(u)$ to be a real, even function. The Dirac delta function is replaced in the commutation relations among the Φ operators, as well as in the bosonic two-point function, by the convolution $g(u-u') = \int du'' f(u-u'')f(u''-u')$. It is thereby softened. We

will choose $f(u)$ so that $g(u) \geq 0$, $g(0) = 1$, and $\epsilon(u)g'(u) < 0$ if $g(u) \neq 0$. To make sure that $g(u)$ goes to 0 very quickly as $|u|$ increases we take it to be a function of u/λ and eventually take the scale parameter λ to zero. This simulates the properties of a Kronecker delta function.

We define the field $\hat{\psi}(x, u) = C(2\pi\alpha)^{-1/2} \exp[i\sqrt{4\pi}\Phi(x, u)]$ which will be the candidate for the canonical fermion field ψ in the appropriate limit. Here C is some constant, to be specified later, that depends on μ and on the choice of $g(u)$. We then find for the fermionic two-point function

$$\langle 0|\hat{\psi}(x, u)\hat{\psi}^\dagger(x', u')|0\rangle = |C|^2 \mu^{[1-g(u,u')]} \frac{1}{2\pi} \frac{1}{[\alpha - i(x - x')]^{g(u-u')}}.$$

As the infra-red cutoff μ goes to zero, this vanishes unless $u = u'$. Choosing C to depend on μ so that C diverges suitably in this limit, we obtain the desired Dirac delta function and the same answer as in the discrete case, with the Kronecker delta replaced by the Dirac delta.

There is a surprise when we consider the four-point function. One must analyze the behavior of the quantity $|C|^4 \mu^{(2+g_{12}+g_{34}-g_{13}-g_{23}-g_{14}-g_{24})}$ as $\mu \to 0$. Here $g_{ij} = g_{(u_i - u_j)}$ with u_1 and u_2 assigned to the fermion fields and u_3 and u_4 to the adjoint fields. In order to obtain the correct behavior when all four internal indices are close we find that we cannot choose $f(u)$ such that $g'(0) = 0$. In fact $g(u)$ must have a cusp at $u = 0$. Thus Gaussian functions for $f(u)$ and $g(u)$ are ruled out. A triangular pulse is acceptable for $g(u)$ and this corresponds to a rectangular pulse for $f(u)$.

Fermion bilinears

We go on to discuss the fermion bilinears. This is where we had trouble in the discrete case. A generic form of such a bilinear operator is $[\psi^\dagger(x_2, u_2), \psi(x_1, u_1)]$ where the arguments are allowed to come together, possibly after various derivatives have been taken. This is the nature of the spatially point-split structure of the charge operator, and of the generators of translations in x, t and u. We proceed to evaluate it using operator-product expansion methods (which are exact in this model). We have

$$[\hat{\psi}_2^\dagger, \hat{\psi}_1] = -\frac{|C|^2}{2\pi} \mu^{(1-g_{12})}(R_{12}{}^{-g_{12}} - R_{21}{}^{-g_{21}}){:}e^{i\sqrt{4\pi}(\Phi_1-\Phi_2)}{:}.$$

Here $R_{12} = \alpha - i(x_1 - x_2)$. The normal ordering is with respect to the bosonic creation and annihilation operators. We introduce $u = \frac{1}{2}(u_1 + u_2)$, $v = u_1 - u_2$, $x = \frac{1}{2}(x_1 + x_2)$ and $\xi = x_1 - x_2$, and obtain

$$[\hat{\psi}_2^\dagger, \hat{\psi}_1] = -\frac{|C|^2}{2\pi} \mu^{[1-g_{12}]}[R(\xi)^{-g(v)} - R(-\xi)^{-g(v)}]\Omega(\xi, v)$$

where $\Omega(\xi, v) = {:}\exp i\sqrt{4\pi}[\Phi(x + \frac{1}{2}\xi, u + \frac{1}{2}v) - \Phi(x - \frac{1}{2}\xi, u - \frac{1}{2}v)]{:}$. Taking v and ξ small, (while keeping $\alpha \ll \xi$), and sending μ to 0 we discover that $[\hat{\psi}_2^\dagger, \hat{\psi}_1] \to -i\delta(v)\Omega(\xi, 0)/(\pi\xi)$.

The chiral charge operator Q is given in terms of the canonical fermions by

$$\hat{Q} = \tfrac{1}{2} \int dx \; du_1 \; du_2 \; \delta(u_1 - u_2)[\psi^\dagger(x, u_2), \psi(x, u_1)].$$

We choose to smooth out the u delta function and replace it by $N_\lambda g(u_1 - u_2)$ where N_λ is chosen so that $N_\lambda \int du_1 \; g(u_1 - u_2) = 1$. Then $N_\lambda g(u_1 - u_2)$ becomes a Dirac delta function as $\lambda \to 0$. We construct the candidate chiral charge $\hat{Q}$ in terms of $\hat{\psi}$ as $\hat{Q} = \lim_{\xi \to 0} N_\lambda \int dx \; du \; dv \; g(v)\tfrac{1}{2}[\hat{\psi}_2^\dagger, \hat{\psi}_1]$ so that

$$Q = \lim_{\xi \to 0} \frac{-iN_\lambda g(0)}{4\pi\xi} \int dx \; du \; [\Omega(\xi, 0) - \Omega(-\xi, 0)] = \frac{N_\lambda}{\sqrt{\pi}} \int dx \; du \; \frac{\partial}{\partial x}\Phi(x, u).$$

The charge density in u space is thus $\hat{Q}(u) = N_\lambda \pi^{-1/2}[\Phi(\infty, 0) - \Phi(-\infty, 0)]$. We then have $[\hat{\psi}(x, u), \hat{Q}(u')] = N_\lambda g(u - u')\hat{\psi}(x, u)$ which becomes the correct canonical relation as $\lambda \to 0$.

The canonical operator $J = -i\tfrac{1}{4} \int dx \; du \; [\psi^\dagger(x, u), \frac{\partial}{\partial u}\psi(x, u)] + (\text{h. c.})$ generates translations in u. It is the analogue of the rotation and boost operators in the tomographic representation of the 3+1 dimensional case. If we were to follow the procedure used for the chiral charge we would find that when expressed in terms of bosonic fields, the candidate generator would vanish. The remedy for this is to back up a step and delay taking μ to zero. Then the spreading function used to define the bilinear, (which need not be the same as what we used for the chiral charge, namely $N_\lambda g(u)$), can be taken to depend on μ in just such a way as to yield the correct answer in terms of canonical boson fields in the limit.

References

[1] S. Coleman, *Phys. Rev. D* **11** (1975) 2088.

[2] S. Mandelstam, *Phys. Rev. D* **11** (1975) 3026.

[3] A. Luther, *Phys. Rev. B* **19** (1979) 320.

[4] C. M. Sommerfield, in "Symmetries in Particle Physics," A. Chodos, I. Bars and C.-H. Tze, eds., Plenum Press (1984) pp. 127-140.

[5] H. Aratyn, *Nuc. Phys. B* **227** (1983) 172.

[6] E. C. Marino, *Phys. Lett.* **B263** (1991) 63.

[7] A. Kovner and P. S. Kurzepa, *Phys. Lett.* **B328** (1994) 506.